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FIBONACCI-TYPE $1D$ AND $2D$ WORDS

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Abstract. Constructing new word sequences with distinctive properties will help us understand infinite words further. In this paper, we study a variant of the one-dimensional ($1D$) infinite Fibonacci word and thus exhibit the possible generations of new aperiodic infinite words with varying characteristics. We also explore ways of constructing Fibonacci-type two-dimensional ($2D$) aperiodic words using continued fractions and Beatty sequences.

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1. Introduction

The study of properties like factor complexity, periodicity, and regularity of infinite words (infinite sequence of characters from some alphabet) is central to combinatorics on words [1]. Combinatorics on words has diversified applications in algorithms related to pattern recognition, text compression, image processing, and DNA computing [2]. The influential work of Axel Thue on repetitions in words (which resulted in the well-known Thue-Morse word) is worth mentioning here. Other noteworthy infinite words are Sturmian, Arnoux–Rauzy, Kolakoski–Oldenburger and Rudin–Shapiro words [3].

Factor complexity of an infinite word w, denoted by $p_w(n)$, $n \geq 1$, is the number of subwords of length n occurring in w. Analysis of factor complexity is significant in understanding the structure of the word.

Another important feature of an infinite word is, periodicity. It is interesting to note that, words that are not periodic (i.e. aperiodic words) influence the design of crystals, tilings and pseudo-random sequences [4, 5].

For aperiodic words, we have, $p_w(n) \geq n+1$ [6]. Notably, Sturmian words have the minimum factor complexity (i.e. $p(n) = n + 1$) among all infinite aperiodic words [1]. Due to this distinctive property, Sturmian words are studied more rigorously in order to generalize various properties of infinite words. In particular, the one-dimensional infinite Fibonacci word is the most sought-after Sturmian word due to its remarkable properties [7]. As the name suggests, Fibonacci words are directly related to the Fibonacci numerical sequence: $F(0) = 1$, $F(1) = 1$, $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$. As presented by Leonardo of Pisa in his book Liber Abaci [8], these numbers are nothing but the number of rabbit pairs (which breed in accordance with some hypothetical assumptions) present n months after we started with a single pair. A two-dimensional (hereafter sometimes written as $2D$) extension of the Fibonacci words also was proposed in [9].

Motivated by these interesting and useful properties of Sturmian words, in this paper, we study a variant of the one-dimensional (hereafter sometimes written as $1D$) infinite Fibonacci word. Also, in two dimensions, we explore two ways of constructing Fibonacci-type $2D$ words, one using continued fractions and the other using Beatty sequences. We understand that, by systematically constructing new word sequences and studying their structural properties, we can generalize many properties of infinite words.

2. Preliminaries

2.1. **Words in Formal Language Theory.** In the theory of formal languages, we have Σ, a finite set of symbols or letters, called an alphabet. By juxtapositioning/concatenating the symbols of Σ we obtain Σ^*, the free monoid generated by Σ. The elements of Σ^* are called words. The empty word, denoted by λ is the neutral element of Σ^*. We also have $\Sigma^+ = \Sigma^* - \{\lambda\}$. If u is a word in Σ^*, $|u|$ denotes the length of u and is the number of letters occurring in u. By definition, $|\lambda| = 0$. For a given word $w \in \Sigma^*$, $u \in \Sigma^*$ is called a prefix (suffix, respectively) of w, if $w = uv$ ($w = vu$, respectively) for some $v \in \Sigma^*$. The reversal of a word $u = a_1a_2\cdots a_n$, $a_i \in \Sigma$, $1 \le i \le n$, is the word $u^R = a_n \cdots a_2a_1$. If $u = u^R$ then u is said to be a palindrome.

A power of a word is defined as the repeated concatenation of the word with itself. That is, u^n is obtained by concatenating u with itself n number of times. A word w is said to be primitive if $w = u^n$ implies $n = 1$ and $w = u$. By Q we denote the set of all primitive words over Σ. To learn more about formal language theory the reader can refer [1].

2.2. **Fibonacci Words: The Familiar Setup.** An analogous setup to Fibonacci numbers is the set of Fibonacci words. For the alphabet $\Sigma = \{a, b\}$, the sequence $\{f_n\}_{n\ge0}$ defined recursively by $f_0 = b$, $f_1 = a$, $f_n = f_{n-1}f_{n-2}$ for $n \ge 2$ is called the Fibonacci words. More explicitly, $f_0 = b, f_1 = a, f_2 = ab, f_3 = aba, f_4 = abaab$ and so on. The set of all Fibonacci words will constitute the Fibonacci language, $F_{a,b}$. That is, $F_{a,b} = \bigcup\limits_{n\ge0} f_n$.

Note that, in the above discussion, we can very well relate b to a *baby* rabbit pair and a to an *adult* rabbit pair, so that the Fibonacci's population growth model of rabbit pairs is established through the symbols a and b. Recall the hypothetical assumptions in the classical setup that (i) no rabbits die (ii) A new baby pair becomes an adult pair one month after birth and breeds a new baby pair every following month. That is, at the end of month 0 (initially), one rabbit pair ($f_0 = b$) is present. At the end of month 1, the *baby* pair becomes an *adult* pair and hence one rabbit pair ($f_1 = a$) is present. At the end of month 2, the *adult* pair breeds a *baby* pair and hence two rabbit pairs ($f_2 = ab$) are present, and so on. Note the interesting relation, $|f_n| = F(n)$ for $n \ge 0$. Also, recall the explicit formula for $F(n)$, i.e., the solution of the Fibonacci recurrence relation, $F(0) = F(1) = 1$, $F(n) = F(n-1) + F(n-2)$, $n \ge 2$,

$$F(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}, \ n \ge 0,$$

which gives the number of rabbit pairs at the end of any month *'n'*.

2.3. **Lindenmayer (L) Systems.** From Section 2.2, one can easily understand the growth process as, "b becomes a" and "a becomes ab" that also simultaneously. These type of languages arising out of parallel rewriting was first studied by Aristid Lindenmayer while exploring the growth processes of multi-cellular organisms [10]. We recall the formal definitions here.

Definition 2.1. [11] *A finite substitution σ over an alphabet Σ is a mapping of Σ^* into the set of all finite nonempty languages (possibly over an alphabet Δ different from Σ) defined as follows. For each letter $a \in \Sigma$, $\sigma(a)$ is a finite nonempty language, $\sigma(\lambda) = \lambda$ and, for all words $w_1, w_2 \in \Sigma^*$, $\sigma(w_1w_2) = \sigma(w_1)\sigma(w_2)$.*

If none of the languages $\sigma(a), a \in \Sigma$, contains the empty word, the substitution σ is referred to as λ-free or non-erasing. If each $\sigma(a)$ consists of a single word, σ is called a morphism. Along these lines, the sequence of Fibonacci words $b, a, ab, aba, \ldots$ can be obtained by iterating the Fibonacci morphism $\sigma : \Sigma \to \Sigma^*$ defined by $\sigma(b) = a, \sigma(a) = ab$, where $\Sigma = \{a, b\}$.

A Lindenmayer system is now defined through a substitution σ.

Definition 2.2. [11] *A $0L$ system is a triple $G = (\Sigma, \sigma, w_0)$, where Σ is an alphabet, σ is a finite substitution on Σ, and w_0 (referred to as the axiom,) is a word over Σ. The $0L$ system is propagating or a $P0L$ system if σ is non-erasing. The $0L$ system G generates the language $L(G) = \{w_0\} \cup \sigma(w_0) \cup \sigma(\sigma(w_0)) \cup \cdots = \bigcup_{i\geq 0} \sigma^i(w_0)$. A $0L$ system (Σ, σ, w_0) is deterministic or a $D0L$ system iff σ is a morphism.*

The system generates the language in a specific order, w_0, $w_1 = \sigma(w_0)$, $w_2 = \sigma^2(w_0)$, $w_3 = \sigma^3(w_0)$ and so on. We will denote the sequence by $S(G)$. The infinite word $\lim_{n\to\infty} \sigma^n(w_0) = w_\infty$ is called the fixed point of the morphism σ. Note that the Fibonacci language $F = \{b, a, ab, aba, abaab, \ldots\}$ (defined through the morphism $\sigma(b) = a, \sigma(a) = ab$,) is a $D0L$ language [12]. In fact, by Definition 2.2, it is a $PD0L$ language.

Let us state a few more related concepts.

Definition 2.3. [11] *An infinite sequence of words $w_i, i \geq 0$, is locally catenative iff, for some positive integers $k, i_1, i_2, \ldots, i_k$, and $q > max(i_1, \ldots, i_k)$, $w_n = w_{n-i_1}w_{n-i_2}\ldots w_{n-i_k}$, whenever $n > q$. A $D0L$ system G is locally catenative iff the sequence $S(G)$ is locally catenative.*

Locally catenative is a desirable property of $D0L$ systems.

Definition 2.4. [11] *For an infinite sequence of words, $w_i, i \geq 0$, the function (from non-negative integers to itself,) $f(n) = |w_n|$ is termed the growth function of the sequence.*

We have the following systematic way of finding the values of $f(n)$ [10, 11].

For a $D0L$ system (Σ, σ, w_0), suppose that the alphabet Σ has k elements, $s_1, s_2, \ldots, s_k$. Let π be the Parikh vector of the axiom. That is, π is the k-dimensional row vector such that, for $i = 1, 2, \ldots, k$, i^{th} component of π equals the number of occurrences of the letter s_i in the axiom w_0. Let the growth matrix M be the k-dimensional square matrix whose $(i, j)^{th}$ entry,

$i, j \in \{1, 2, \ldots, k\}$, equals the number of occurrences of a_j, in $\sigma(a_i)$. Then, the values of the growth function are obtained by $f(n) = \pi M^n \eta$, $n \geq 0$, where η is the k-dimensional column vector with all components equal to 1.

In fact, M is the arithmetization of the morphism σ and is sometimes called as the matrix of the morphism. Note that, for $i = 1, 2, \ldots, k$, the i^{th} row of M is the Parikh vector of $\sigma(a_i)$. Hence, the i^{th} row of M^n will be the Parikh vector of $\sigma^n(a_i)$ [3]. Hence, for an axiom $w_0 = a_i, i \in \{1, 2, \ldots, k\}$, the Parikh vector of $\sigma^n(w_0)$ can be directly obtained from M^n.

Another notion worth mentioning is *recurrence systems*. It is a formal framework to overcome the ambiguity of the English language while describing a developmental language [10]. Also it is a nondeterministic generalization of locally catenative D0L systems [12].

Definition 2.5. [10] *A recurrence system is a* 6*-tuple* $S = (\Sigma, \Omega, d, \mathscr{A}, \mathscr{F}, \omega)$, *where (1)* Σ *is a finite non-empty set of symbols (the alphabet), (2)* $\Omega = \mathbb{N}^w$ *(*w *is called the width of* S*) is a finite non-empty set (the index set), (3)* d *is a positive integer (the depth of* S*), (4)* $\mathscr{A}$ *is a function, associating with each* $(x, y) \in \Omega \times \mathbb{N}^d$ *a finite set* $A_{x,y}$ *(of axioms) such that* $A_{x,y} \subset \Sigma^*$*, (5)* $\mathscr{F}$ *is a function, associating with each* $x \in \Omega$ *a non-empty finite set* F_x *(of recurrence formulas), such that* $F_x \subset ((\Omega \times \mathbb{N}^d) \cup \Sigma)^*$*, (6)* $\omega \in \Omega$ *(the distinguishable index).*

A recurrence system with width w and depth d is called a (w, d) recurrence system. If $A_{x,y}$ for all x, y has exactly one rule or empty then the recurrence system will be deterministic. Languages generated by recurrence systems are said to be recurrence languages. Locally catenative systems are nothing but deterministic recurrence systems of width 1 [10, 11].

2.4. **Two-dimensional Words.** The concepts of formal language theory can be extended to two dimensions [13]. A $2D$ word (otherwise called a picture or array) is a rectangular arrangement of symbols taken from Σ.

Definition 2.6. [14] *A* $2D$ *word* $u = [u_{i,j}]_{1 \leq i \leq m, 1 \leq j \leq n}$ *of size* (m, n) *over* Σ *is a finite* $2D$ *rectangular arrangement of letters from* Σ*, as shown below.*

$$u = \begin{matrix} u_{1,1} & u_{1,2} & \cdots & u_{1,n-1} & u_{1,n} \\ u_{2,1} & u_{2,2} & \cdots & u_{2,n-1} & u_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{m-1,1} & u_{m-1,2} & \cdots & u_{m-1,n-1} & u_{m-1,n} \\ u_{m,1} & u_{m,2} & \cdots & u_{m,n-1} & u_{m,n} \end{matrix}.$$

The number of rows and columns of u are denoted by $|u|_{\text{row}}$ and $|u|_{\text{col}}$, respectively. The empty array, denoted by Λ is the array of size $(0, 0)$ and the arrays of sizes $(m, 0)$,$(0, m)$ for $m > 0$ are not defined. The set of all arrays over Σ including Λ, is denoted by Σ^{**} and Σ^{++} will denote the set of all non-empty arrays over Σ. Any subset of Σ^{**} is called a picture language.

Given an array u, the set of coordinates $\{1, 2, \ldots, |u|_{\text{row}}\} \times \{1, 2, \ldots, |u|_{\text{col}}\}$ constitute the domain of u. A subarray (subword or factor) of u is denoted by $u[(i, j), (i', j')]$ and is the portion of u located in the domain $\{i, i+1, \ldots, i'\} \times \{j, j+1, \ldots, j'\}$, where $1 \leq i \leq i' \leq |u|_{\text{row}}, 1 \leq j \leq j' \leq |u|_{\text{col}}$.

Similar to the concatenation operation on Σ^*, we have the column concatenation and the row concatenation operations on Σ^{**}.

Definition 2.7. [13] *Let* $m_1, n_1, m_2, n_2 > 0$ *and let* $u = [u_{i,j}]_{1\leq i\leq m_1, 1\leq j\leq n_1}$, $v = [v_{i,j}]_{1\leq i\leq m_2, 1\leq j\leq n_2}$ *be arrays (over* Σ*) of sizes* (m_1, n_1) *and* (m_2, n_2)*, respectively. Then the column concatenation of* u *and* v*, denoted by* $u \oplus v$*, is a partial operation, defined only if* $m_1 = m_2 = m$*. Similarly, the row concatenation of* u *and* v*, denoted by* $u \ominus v$*, is also a partial operation, defined only if* $n_1 = n_2 = n$*. They are defined as given below.*

$$u \oplus v = \begin{matrix} u_{1,1} & \cdots & u_{1,n_1} & v_{1,1} & \cdots & v_{1,n_2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ u_{m,1} & \cdots & u_{m,n_1} & v_{m,1} & \cdots & v_{m,n_2} \end{matrix}, \quad u \ominus v = \begin{matrix} u_{1,1} & \cdots & u_{1,n} \\ \vdots & \ddots & \vdots \\ u_{m_1,1} & \cdots & u_{m_1,n} \\ v_{1,1} & \cdots & v_{1,n} \\ \vdots & \ddots & \vdots \\ v_{m_2,1} & \cdots & v_{m_2,n} \end{matrix}.$$

The column,row concatenations of u *and the empty array* Λ *are always defined and* Λ *is a neutral element for both the operations.*

For a $u \in \Sigma^{**}$, an array $v \in \Sigma^{**}$ is said to be a prefix of u (suffix of u, respectively), if $u = (v \ominus x) \oplus y$ ($u = y \oplus (x \ominus v)$, respectively) for some $x, y \in \Sigma^{**}$. If $x \in \Sigma^{++}$, then by $(x^{k_1\oplus})^{k_2\ominus}$ we mean the array constructed by repeating x, k_1 times column-wise to get $x^{k_1\oplus}$ and then repeating $x^{k_1\oplus}$, k_2 times row-wise. An array $w \in \Sigma^{++}$ is said to be 2D *primitive* if $w = (x^{k_1\oplus})^{k_2\ominus}$ implies that $k_1 k_2 = 1$ and $w = x$.

2.5. **Two-dimensional Fibonacci Words.** The extension of $1D$ Fibonacci words to Fibonacci arrays is defined as below.

Definition 2.8. [9] *Let* $\Sigma = \{a, b, c, d\}$*. The sequence of Fibonacci arrays,* $\{f_{m,n}\}$ *where* $m, n \geq 0$*, is defined as:*

(1) $f_{0,0} = \beta, f_{0,1} = \gamma, f_{1,0} = \delta, f_{1,1} = \alpha$ *where* α, β, γ *and* δ *are symbols from* Σ *with some but not all, among* α, β, γ *and* δ *might be identical.*
(2) *For* $k \geq 0, m, n \geq 1$, $f_{k,n+1} = f_{k,n} \oplus f_{k,n-1}$, $f_{m+1,k} = f_{m,k} \ominus f_{m-1,k}$.

In this paper, we fix $f_{0,0} = a, f_{0,1} = b, f_{1,0} = c, f_{1,1} = d$ and we call $f_{k,n+1} = f_{k,n} \oplus f_{k,n-1}$ as column-wise expansion and $f_{m+1,k} = f_{m,k} \ominus f_{m-1,k}$ as row-wise expansion. In Example 2.9 we construct $f_{2,2}$ in both ways.

Example 2.9. *By applying row-wise expansion first, and then column-wise expansion, we have,* $f_{2,2} = f_{1,2} \ominus f_{0,2} = (f_{1,1} \oplus f_{1,0}) \ominus (f_{0,1} \oplus f_{0,0})$.

By applying column-wise expansion first, and then row-wise expansion, we will have, $f_{2,2} = f_{2,1} \oplus f_{2,0} = (f_{1,1} \ominus f_{0,1}) \oplus (f_{1,0} \ominus f_{0,0})$.

Since, $f_{0,0} = a, f_{0,1} = b, f_{1,0} = c, f_{1,1} = d$*, we can write,* $f_{2,2} = \begin{matrix} d & c \\ b & a \end{matrix}$.

For a more detailed study on Fibonacci arrays [14] or [15] can be referred.

3. New Aperiodic $1D$ Words

In this section we present the generation of a new aperiodic $1D$ infinite word. Unlike the classical Fibonacci rabbit population growth model, in this variant, an *adult* pair produces 'r' *baby* pairs, every month. Note that in the familiar (classical) setup, $r = 1$. We have the corresponding $D0L$ system as: $G = (\Sigma, \sigma, f_0)$, where,

$$\Sigma = \{a, b\}, \quad \sigma : b \to a,\ a \to ab^r, \quad f_0 = b.$$

TABLE 1. Growth of rabbit population when r baby pairs are bred

End of Month n	**Rabbit Pairs Present**	**Number of Rabbit Pairs**
$n = 0$	b	1
$n = 1$	a	1
$n = 2$	ab^r	$r + 1$
$n = 3$	$ab^r a^r$	$2r + 1$
$n = 4$	$ab^r a^r (ab^r)^r$	$2r^2 + 3r + 1$

TABLE 1 gives a glimpse of the population growth. The system is a deterministic $(1, 2)$ recurrence system and is locally catenative. It is easy to form the recurrence relation for this sequence of words as:

$$f_0 = b, f_1 = a, \text{and for } n \geq 2, f_n = f_{n-1}(f_{n-2})^r.$$

The formal way of finding the recurrence relation is as below.

$$\begin{aligned}\sigma^n(b) &= \sigma^{n-1}(a) = \sigma^{n-2}(ab^r) = \sigma^{n-2}(a)\,\sigma^{n-2}(b^r)\\ &= \sigma^{n-2}(\sigma(b))\ \left(\sigma^{n-2}(b)\right)^r = \sigma^{n-1}(b)\ \left(\sigma^{n-2}(b)\right)^r\end{aligned}$$

and hence $f_n = f_{n-1}(f_{n-2})^r$.

Now, the recurrence relation for the growth function, $f(n) = |f_n|$ will be,

$$f(0) = 1, f(1) = 1, \text{ and for } n \geq 2, f(n) = f(n-1) + rf(n-2).$$

Solving the recurrence relation, we get, for $n \geq 0$,

$$f(n) = \frac{1}{\sqrt{1+4r}} \left(\frac{1+\sqrt{1+4r}}{2}\right)^{n+1} - \frac{1}{\sqrt{1+4r}} \left(\frac{1-\sqrt{1+4r}}{2}\right)^{n+1}.$$

Since the system is a locally catenative $D0L$ system, the growth function can also be given by,

$$f(n) = \pi M^n \eta,\ n \geq 0, \text{ where } \pi = (0\,1),\ M = \begin{pmatrix} 1 & r \\ 1 & 0 \end{pmatrix},\ \eta = (1\,1)^T.$$

Further, as the axiom is $w_0 = b$, the second row of M^n will give $|f_n|_a$, the number of a's in $|f_n|$, and $|f_n|_b$, the number of b's in $|f_n|$. It is obvious that $|f_n|_a + |f_n|_b = f(n)$.

Now, let us state and prove a few properties of the Fibonacci language associated with this variant. We have the language associated with this variant as $F_{a,b}^{r\text{-}babies} = \{f_0, f_1, f_2, \ldots\} = \{b, a, ab^r, ab^r a^r, ab^r a^r (ab^r)^r, \ldots\}$.

Example 3.1. *When $r = 3$ we have,*

$f_0 = b,\ f_1 = a,\ f_2 = ab^3 = abbb,\ f_3 = ab^3a^3 = abbbaaa$

$f_4 = ab^3a^3(ab^3)^3 = abbbaaaabbbabbbabbb$

$f_5 = ab^3a^3(ab^3)^3(ab^3a^3)^3 = abbbaaaabbbabbbabbbabbbaaaabbbaaaabbbaaa$

First let us analyse the primitive nature of $F_{a,b}^{r\text{-}babies}$. We need the following theorems.

Theorem 3.2. [16] *If $p, q \in Q$, then $p^i q^j \in Q$ for all $i, j \geq 2$.*

Theorem 3.3. [17] *If $u = u_1u_2 \neq \epsilon$ and $u' = u_2u_1$, then u is primitive if and only if u' is primitive.*

Theorem 3.4. *The language $F_{a,b}^{r\text{-}babies}$ is primitive. That is each word in the language is primitive.*

Proof. Clearly, f_0, f_1, f_2 are primitive. For $n \geq 3$, we have $f_n = f_{n-1}f_{n-2}^r = f_{n-2}f_{n-3}^r f_{n-2}^r$. Consider $f_n' = f_{n-2}^{r+1}f_{n-3}^r$, a conjugate of f_n. As f_{n-2} and f_{n-3} are primitive, by Theorem 3.2, f_n' is primitive. Hence, by Theorem 3.3, f_n is primitive . □

As $F_{a,b}^{r\text{-}babies}$ is primitive, we analyse the quasiperiodic nature of the fixed point of the morphism σ. Note that, the fixed point of σ is the infinite word, $f_\infty^{r\text{-}babies} = ab^ra^r(ab^r)^r(ab^ra^r)^r \cdots = ab^ra^{r+1}b^rab^rab^r \cdots ab^ra^{r+1} \cdots$.

We recall the concept of quasiperiodicity from [18]. We say that a string w covers another string z if every position of z is covered by some occurrence of w in z. If z is covered by $w \neq z$, we say that z is quasiperiodic and w is called a quasiperiod of z. A string z which is not quasiperiodic is called superprimitive.

Theorem 3.5. *$f_\infty^{r\text{-}babies}$ is superprimitive.*

Proof. We observe that 'b' always occurs in r^{th} power in $f_\infty^{r\text{-}babies}$, sandwiched between powers of 'a'. The four type of patterns that occur in $f_\infty^{r\text{-}babies}$ are, $ab^ra^{r+1}b^r$, $a^{r+1}b^rab^r$, ab^rab^r and $a^{r+1}b^ra^{r+1}b^r$. Since ab^ra^{r+1} is a prefix of $f_\infty^{r\text{-}babies}$, the possible choices for a quasiperiod q of $f_\infty^{r\text{-}babies}$ are ab^r,ab^ra,ab^ra^{r+1}. Note that ab^r and ab^ra are obviously ruled out and the other choice $q = ab^ra^{r+1}$ cannot cover segments of the pattern ab^rab^r. Hence $f_\infty^{r\text{-}babies}$ is superprimitive. □

Recall that, for $n \in \mathbb{N}$, $p_w(n)$, the factor (or subword) complexity of a word w, counts the number of subwords of length n occurring in w. It is a measure that analyses the amount of randomness present in w and hence helps us understand the structure of w. It is proved that the factor complexity of periodic words is bounded and for aperiodic words $p(n) \geq n + 1$ [6].

Let us now, investigate the factor complexity of $f_\infty^{r\text{-}babies}$. Recall that a factor u of length n of an infinite word over a binary alphabet $\{a, b\}$ is right special if both ua and ub are factors of the infinite word.

Theorem 3.6. *Let $p(n)$ be the complexity function of $f_\infty^{r\text{-}babies}$. Then, for $n \geq r + 1$ the first finite differences of $p(n)$ (i.e., $p(n + 1) - p(n)$) is 1.*

Proof. As noted in the proof of Theorem 3.5, 'b' always occur in r^{th} power in $f_\infty^{r\text{-}babies}$. Also, '$a$' occurs either in its first power or in its $(r+1)^{th}$ power. So, for $n \geq r+1$, a subword of $f_\infty^{r\text{-}babies}$ of length n will be right special only if it ends with 'ba'. This is because, in any subword of $f_\infty^{r\text{-}babies}$, '$b$' has to complete a run of length r and so, to the right of a subword ending with 'b' we can affix either 'a' or 'b' and not both. Similarly, to the right of a subword ending with aa we can affix either 'a' or 'b' and not both. So, the subword ending with ba only will be right special and hence $p(n+1) = p(n) + 1$. □

4. New Aperiodic 2D Words Using Continued Fractions

First let us recall the required definitions.

Definition 4.1. [19] *A finite continued fraction is an expression of the form*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \frac{1}{a_n}}}$$

which we abbreviate as $[a_0; a_1, \ldots, a_n]$*. The continued fraction is called simple, if the* a_i*'s are integers such that* $a_i \geq 1$ *for* $1 \leq i \leq n$*.*

In the continued fraction, a_0 is called the integer part and $a_1, \ldots, a_n$ are called the partial quotients. The value of a simple continued fraction is rational if and only if the continued fraction is finite. If $a_{n+t} = a_n$ for all $n > N$, we write $[a_0; a_1, a_2, \ldots] = [a_0; a_1, \ldots, a_N, \overline{a_{N+1}, a_{N+2}, \ldots, a_{N+t}}]$ and say the continued fraction is ultimately periodic. For examples, we have $\frac{12}{65} = [0; 5, 2, 2, 2]$, $\sqrt{23} = [4; 1, 3, 1, 8, 1, 3, 1, 8, \ldots] = [4; \overline{1, 3, 1, 8}]$.

Theorem 4.2. [19] *The partial quotients in the continued fraction expansion for real* α *are ultimately periodic if and only if* α *is an irrational number that satisfies a quadratic equation with integer coefficients.*

The following definition (Definition 4.3) is one among the various equivalent definitions of 1D Sturmian words and has motivated us to construct Fibonacci-type 2D words using continued fractions.

Definition 4.3. [20] *Given the continued fraction* $[0; a_1 + 1, a_2, a_3, \ldots]$ *of an irrational* $\alpha \in (0, 1)$*, define the sequence of words* $(s_n)_{n \geq -1}$ *by*

$$s_{-1} = 1,\ s_0 = 0,\ s_n = s_{n-1}^{a_n} s_{n-2},\ n \geq 1.$$

This sequence of words is called the standard sequence of α*. The infinite word* $c_\alpha = \lim_{n \to \infty} s_n$ *is called the characteristic word of* α*.*

4.1. **Two-dimensional Characteristic Words.**

A class of 2D words (over a three-letter alphabet) obtained by coding discrete planes was studied by Arnoux [21] and Vuillon [22]. Prior to this, Cassaigne characterized 2D words with rectangle complexity $mn + 1$ [23].

In the following, we construct aperiodic Fibonacci-type 2D words (or, say in general, Sturmian-type 2D words) using the continued fraction expansions of two irrational numbers $\alpha_1, \alpha_2 \in (0, 1)$.

Definition 4.4. *Let the continued fraction expansions of the irrationals* $\alpha_1, \alpha_2 \in (0, 1)$ *be* $[0; d_1 + 1, d_2, d_3, \ldots]$ *and* $[0; e_1 + 1, e_2, e_3, \ldots]$*, respectively. Define a sequence of* 2D *words* $\{s_{m,n}\}_{m,n \geq -1}$ *recursively as follows.*

With $s_{-1,-1} = a$, $s_{-1,0} = b$, $s_{0,-1} = c$, $s_{0,0} = d$*, for* $k \geq -1$ *and* $m, n \geq 1$*,*

$$s_{m,k} = s_{m-1,k}^{d_m \ominus} \ominus s_{m-2,k}, \quad s_{k,n} = s_{k,n-1}^{e_n \varobar} \varobar s_{k,n-2}. \tag{1}$$

Then the sequence $\{s_{m,n}\}_{m,n \geq -1}$ *is called the* 2D *standard sequence of* (α_1, α_2) *over the alphabet* $\Sigma = \{a, b, c, d\}$*. Since* α_1, α_2 *are irrationals,*

$$\lim_{m \to \infty} \lim_{n \to \infty} s_{m,n} = c_{\alpha_1, \alpha_2}$$

is an infinite 2D *word and is called the characteristic word of* (α_1, α_2)*.*

The following example illustrates Definition 4.4 by generating a finite $2D$ word.

Example 4.5. *For* $\alpha_1 = \frac{1}{\sqrt{7}}$ *and* $\alpha_2 = \frac{1}{\sqrt{11}}$*, we have,*

$$\frac{1}{\sqrt{7}} = [0; 2, \overline{1, 1, 1, 4}] \text{ and } \frac{1}{\sqrt{11}} = [0; 3, \overline{3, 6}].$$

Let us find $s_{3,2}$ *by taking* $m = 3, n = 2$*. Using (1), we have,*

$$\begin{aligned} s_{3,2} &= s_{2,2}^{1\ominus} \ominus s_{1,2} = (s_{1,2}^{1\ominus} \ominus s_{0,2})^{1\ominus} \ominus s_{1,2} \\ &= ((s_{1,1}^{3\oplus} \oplus s_{1,0})^{1\ominus} \ominus (s_{0,1}^{3\oplus} \oplus s_{0,0}))^{1\ominus} \ominus (s_{1,1}^{3\oplus} \oplus s_{1,0}). \end{aligned}$$

Let $s_{-1,-1} = a$*,* $s_{-1,0} = b$*,* $s_{0,-1} = c$*,* $s_{0,0} = d$*. Using (1), we get the words* $s_{0,1}$*,* $s_{1,0}$ *and* $s_{1,1}$ *as* $s_{0,1} = d\ d\ c$*,* $s_{1,0} = \begin{matrix} d \\ b \end{matrix}$ *and* $s_{1,1} = \begin{matrix} d & d & c \\ b & b & a \end{matrix}$*. Therefore,*

$$s_{3,2} = \begin{matrix} d & d & c & d & d & c & d & d & c & d \\ b & b & a & b & b & a & b & b & a & b \\ d & d & c & d & d & c & d & d & c & d \\ d & d & c & d & d & c & d & d & c & d \\ b & b & a & b & b & a & b & b & a & b \end{matrix}.$$

Recall that the limiting case $(m, n \to \infty)$ will produce the infinite $2D$ word. It is to be noted that to obtain the limiting case of the sequence $\{s_{m,n}\}$, we have to analyse the pattern governing the finite words of the sequence. For this the periodicity of the continued fraction expressions can be used.

Remark 4.6. *The standard sequence can be alternately defined using a morphism* ϕ*, similar to the discussion in* [24]*. But depending on the partial quotients of* α_1 *and* α_2*, the size of the images* $\phi(a), \phi(b), \phi(c), \phi(d)$ *may become large, making the work tiresome.*

Remark 4.7. *The* $2D$ *Fibonacci word is the characteristic word of* $(\alpha_1,\alpha_2) = (\frac{1}{\varphi}, \frac{1}{\varphi})$*, where* φ *is the golden ratio.*

We end the section by validating the construction given in Definition 4.4.

Theorem 4.8. *The* $2D$ *infinite word* c_{α_1,α_2}*, constructed in Definition 4.4, is aperiodic.*

Proof. Suppose that c_{α_1,α_2} is periodic. Then the periodic nature might have occurred in any one of the two directions or in both. Assume that it has occurred along the direction of the columns, i.e., 'left to right'. That is $c_{\alpha_1,\alpha_2} = \lim_{k\to\infty} u^{k\oplus}$ for some $u \in \Sigma^{**}$. But from (1) (by doing column-wise expansion first), it is clear that each row of $s_{m,n}$ and hence each row of c_{α_1,α_2} is a $1D$ Sturmian word and are aperiodic. This is a contradiction to the assumption that c_{α_1,α_2} exhibits periodic nature along the direction of the columns. Hence, $s_{m,n}$ and thus c_{α_1,α_2} is aperiodic. The arguments of the proofs in the other cases where periodicity occurs along the direction of rows (i.e., 'top to bottom') or along both directions are similar. □

5. New aperiodic 2D words using Beatty Sequences

In this section we give another insight into the construction of aperiodic infinite $2D$ words using Beatty sequences. This can be considered as an alternative way to the construction discussed in Section 4.

First, let us recall some basics of Beatty sequences.

Definition 5.1. [25] *A Beatty sequence is a set $B = \{\lfloor rn \rfloor : n \geq 1\}$ for some irrational number, $r > 1$. Two Beatty sequences B and B' are complementary if B and B' form a partition of $\mathbb{N} = \{1, 2, 3, \ldots\}$.*

Lemma 5.2. [25] *The Beatty sequences $B = \{\lfloor rn \rfloor : n \geq 1\}$ and $B' = \{\lfloor r'n \rfloor : n \geq 1\}$ are complementary iff $\frac{1}{r} + \frac{1}{r'} = 1$.*

Given below is an example of complementary Beatty Sequences.

Example 5.3. *Let $r = \sqrt{3}$ and $r' = \frac{3+\sqrt{3}}{2}$. Then, we get the Beatty sequences $B = \{1, 3, 5, 6, 8, 10 \ldots\}$ for r, and $B' = \{2, 4, 7, 9, 11, \ldots\}$ for r'. As $\frac{1}{r} + \frac{1}{r'} = 1$, B and B' are complementary. We say, B and B' are the complementary Beatty sequences generated by r.*

Let B and B' be the complementary Beatty sequences generated by $r = \frac{1+\sqrt{5}}{2}$, the *golden ratio.* Then, we know that, in the infinite Fibonacci word $f_\infty = abaababa\cdots$, the i^{th} letter will be a, if $i \in B$ and b, if $i \in B'$ [7]. Further, if we wish to construct the n^{th} Fibonacci word f_n, we can consider the construction of the prefix of length $F(n)$ of f_∞. Hence, to construct f_n directly from B and B', we consider the subsets B_s and B'_s respectively of B and B' such that B_s and B'_s form a partition of $\{1, 2, \ldots, F(n)\}$. Then, the i^{th} letter of f_n will be a, if $i \in B_s$ and b, if $i \in B'_s$. Note that different values of r generate different Sturmian words.

A natural extension of the aforementioned construction of $1D$ Fibonacci words (Sturmian words, in general) to two dimensions is considered in the following by taking a pair of complementary Beatty sequences.

Theorem 5.4. *Let $r_1, r_2 > 0$ be two irrational numbers. Let B and B' be the complementary Beatty sequences generated by r_1. Let C and C' be the complementary Beatty sequences generated by r_2. Let Σ be an alphabet and let $f_{0,0}, f_{0,1}, f_{1,0}, f_{1,1} \in \Sigma$ with all of them different. Then, the 2D word $w^{r_1,r_2}_{\infty,\infty}$ with its*

$$(i, j)^{th} \ entry = \begin{cases} f_{1,1}, & \text{if (i,j)} \in B \times C \\ f_{1,0}, & \text{if (i,j)} \in B \times C' \\ f_{0,1}, & \text{if (i,j)} \in B' \times C \\ f_{0,0}, & \text{if (i,j)} \in B' \times C' \end{cases}$$

will be aperiodic.

Proof. Since r_1 and r_2 are irrational, the infinite words present in each row and in each column of $w^{r_1,r_2}_{\infty,\infty}$ are $1D$ Sturmian words and are aperiodic. This can be understood by observing that every row is generated through the complementary Beatty sequences C and C' and every column is generated through the complementary Beatty sequences B and B' . Since, every row and every column of $w^{r_1,r_2}_{\infty,\infty}$ are aperiodic, $w^{r_1,r_2}_{\infty,\infty}$ itself is aperiodic. $\square$

For an easier understanding, let us construct a $2D$ finite word of size $(4,4)$ using the Beatty sequences of $\sqrt{3}$ and $\sqrt{2}$. Note that, when we consider finite subsets of B, B', C, C', we get finite subwords of the infinite word $w_{\infty,\infty}^{r_1;r_2}$, and when we consider all of B, B', C, C', we get the infinite word $w_{\infty,\infty}^{r_1;r_2}$.

Example 5.5. *For $r_1 = \sqrt{3}$, we get the complementary Beatty sequences $B = \{1,3,5,6,8,10,\ldots\}$ and $B' = \{2,4,7,9,11,14,\ldots\}$. For $r_2 = \sqrt{2}$, we get the complementary Beatty sequences $C = \{1,2,4,5,7,\ldots\}$ and $C' = \{3,6,10,13,17,\ldots\}$. Let us construct a finite $2D$ word of size $(4,4)$ using the subsets, $B_s = \{1,3\} \subset B$, $B'_s = \{2,4\} \subset B'$, $C_s = \{1,2,4\} \subset C$, $C'_s = \{3\} \subset C'$. Note that as we are constructing a $2D$ word of size $(4,4)$, we have selected B_s, B'_s, C_s, C'_s such that $|B_s \cup B'_s| = 4$ and $|C_s \cup C'_s| = 4$.*

Now, the required Cartesian products are,
$B_s \times C_s = \{(1,1),(1,2),(1,4),(3,1),(3,2),(3,4)\}$, $B_s \times C'_s = \{(1,3),(3,3)\}$,
$B'_s \times C_s = \{(2,1),(2,2),(2,4),(4,1),(4,2),(4,4)\}$, $B'_s \times C'_s = \{(2,3),(4,3)\}$.
Let $\Sigma = \{a,b,c,d\}$ and fix $f_{0,0} = a, f_{0,1} = b, f_{1,0} = c, f_{1,1} = d$. Then by Theorem 5.4, we get the entries of w as below.

$$w = \begin{array}{|cccc|}\hline f_{1,1} & f_{1,1} & f_{1,0} & f_{1,1} \\ f_{0,1} & f_{0,1} & f_{0,0} & f_{0,1} \\ f_{1,1} & f_{1,1} & f_{1,0} & f_{1,1} \\ f_{0,1} & f_{0,1} & f_{0,0} & f_{0,1} \\ \hline \end{array} = \begin{array}{|cccc|}\hline d & d & c & d \\ b & b & a & b \\ d & d & c & d \\ b & b & a & b \\ \hline \end{array}.$$

Remark 5.6. *If we take both r_1, r_2 as the golden ratio $\frac{1+\sqrt{5}}{2} = r$(say), then $w_{\infty,\infty}^{r,r}$ will be the $2D$ infinite Fibonacci word.*

Example 5.7 exhibits the construction of a $2D$ finite Fibonacci word.

Example 5.7. *Let us construct $f_{4,3}$. As $F(4) = 5$, $F(3) = 3$, we take, $B_s = \{1,3,4\}$, $B'_s = \{2,5\}$, $C_s = \{1,3\}$, $C'_s = \{2\}$ to get the following.*
$B_s \times C_s = \{(1,1),(1,3),(3,1),(3,3),(4,1),(4,3)\}$,
$B_s \times C'_s = \{(1,2),(3,2),(4,2)\}$, $B'_s \times C_s = \{(2,1),(2,3),(5,1),(5,3)\}$,
$B'_s \times C'_s = \{(2,2),(5,2)\}$.
Letting $f_{0,0} = a$, $f_{0,1} = b, f_{1,0} = c, f_{1,1} = d$, by Theorem 5.4, we get,

$$f_{4,3} = \begin{array}{|ccc|}\hline f_{1,1} & f_{1,0} & f_{1,1} \\ f_{0,1} & f_{0,0} & f_{0,1} \\ f_{1,1} & f_{1,0} & f_{1,1} \\ f_{1,1} & f_{1,0} & f_{1,1} \\ f_{0,1} & f_{0,0} & f_{0,1} \\ \hline \end{array} = \begin{array}{|ccc|}\hline d & c & d \\ b & a & b \\ d & c & d \\ d & c & d \\ b & a & b \\ \hline \end{array}.$$

6. Conclusions

In this paper we have generated and examined a new $1D$ aperiodic infinite word by varying an assumption in the classical rabbit growth model of Fibonacci. Further, using continued fraction expansions and Beatty sequences we have discussed ways of generating new $2D$ aperiodic words which are similar to $2D$ Fibonacci/Sturmian words. Future direction might be towards studying further variants of the $1D/2D$ infinite Fibonacci word so as to generate few more new $1D/2D$ aperiodic infinite words.

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